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Hilbert spaces

Let X be a vector space over $\mathbb{C}$.

An inner product on X $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ satisfies
- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle x, x \rangle \geq 0$ (so it's real)
- $\langle x, x \rangle = 0 \iff x = 0$.

Examples:
- $\mathbb{C}$ with $\langle z, w \rangle = z\overline{w}$
- $\mathbb{C}^n$ with $\langle x, y \rangle = y^H x$ (H – conjugate transpose)
- ℓ^2 sequence space $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$.
  - ↳ Remember Hölder and Minkowski inequality
- $C([a,b], \mathbb{C})$ with
$$\langle f, g \rangle = \int_a^b f(x)\, \overline{g(x)}\, dx$$

Th^m (Cauchy–Schwartz inequality)

$$\boxed{|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}}$$

Corollary: $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on X.

Proof:
- $\|x\| \geq 0$
- $\|x\| = 0 \iff \langle x, x \rangle = 0 \iff x = 0$
- $\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{\lambda \overline{\lambda} \langle x, x \rangle} = |\lambda| \sqrt{\langle x, x \rangle} = |\lambda| \, \|x\|$

Triangle inequality
$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2 \\ &= \|x\|^2 + 2\,\mathrm{Re}\langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$
□

- Norm → Metric $d(x,y) = \|x - y\|$
- Topology & convergence

Any Cauchy sequence converges

A Hilbert space is a complete inner product space

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Example : ℓ^2 is a Hilbert space.

Proof : Let $(x^{(n)})$ be a Cauchy sequence in ℓ^2

$\Rightarrow \forall \varepsilon > 0 \ \exists N \in \mathbb{N} : n, m \geq N \Rightarrow \|x^{(n)} - x^{(m)}\|_2 \leq \varepsilon.$

$\forall k \in \mathbb{N} \quad \underbrace{|x_k^{(n)} - x_k^{(m)}|^2}_{\text{Cauchy in } \mathbb{C}.} \leq \sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^2 \leq \varepsilon^2$

$\Rightarrow x_k^{(n)} \xrightarrow{n \to \infty} x_k$ gives a sequence x.

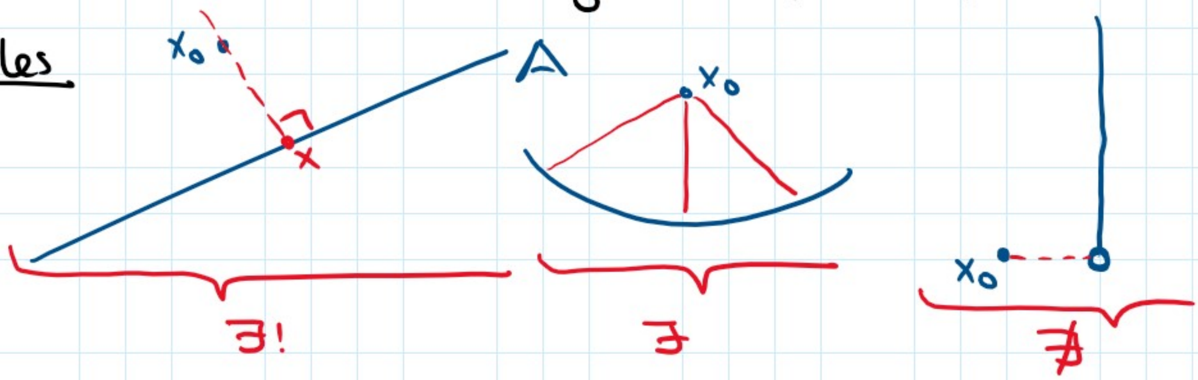
Claim : $x^{(n)} \longrightarrow x$ in norm and $x \in \ell^2$.

$\boxed{\text{Orthogonality}}$ Best approximation

$A \subseteq X$ nonempty. $\boxed{\delta(x_0, A) = \inf_{x \in A} \|x - x_0\|}$

Does there exist $x \in A$ realizing the infimum ?

Examples



Th^m : If A is nonempty, closed convex Then $\exists!$ best approximation.

Corollary : $V \subseteq X$ closed linear subspace $\Rightarrow \exists!$ —〃—

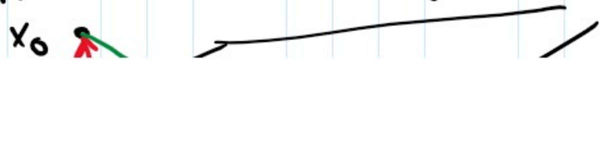
In the above case, we can use orthogonal projection.

- $V^\perp = \{ x \in X : \langle x, y \rangle = 0 \ \forall y \in V \}$.

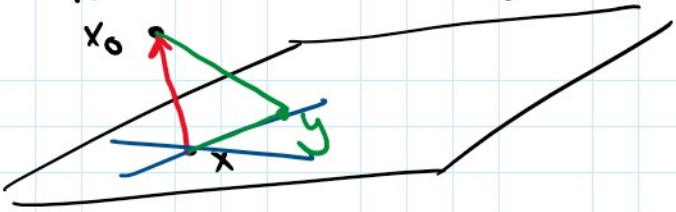
orthogonal complement.

- $V^\perp$ is always a closed linear subspace.
- $(V^\perp)^\perp = \overline{\text{span}}(V) \supseteq V$
- If V is a closed linear subspace, $\boxed{X = V \oplus V^\perp}$

Th^m $V \subseteq X$ closed linear subspace. $x_0 \in X$. The best approximation satisfies $x - x_0 \in V^\perp$. $(x_0 \in V)$.



approximation satisfies $x - x_0 \in V^{\perp}$. $(x_0 \in X)$.



Proof: if $x - x_0 \in V^{\perp}$, then $\forall y \in V$, we have

$$\|y - x_0\|^2 = \|(y-x) + (x - x_0)\|^2 = \|y-x\|^2 + \|x - x_0\|^2$$
$$\geq \|x - x_0\|^2$$

$$\delta(x_0, V) = \inf_{y \in V} \|y - x_0\| \geq \|x - x_0\| \quad (\text{equality for } x \in V) \qquad \square$$

- Converse statement see lecture notes.

Orthogonal projection Th^{m}

$V \subseteq X$ closed linear subspace. $x_0 \in X$, Px_0 is the unique point with $x_0 - Px_0 \in V^{\perp}$. Then
- $x \mapsto Px$ is linear
- $\|Px\| \leq \|x\|$
- $P^2 = P$ (P is a projection)
- $\ker(P) = V^{\perp}$, $\operatorname{ran}(P) = V$

Proof: Exercise $\square$

Bounded Linear maps

$T : X \to Y$ is a linear map.

We say T is bounded if $\boxed{\|Tx\| \leq C \cdot \|x\|}$ for some $C > 0$ (*)

Rm^{k}: Not $\|Tx\| \leq M$. This is impossible for (nontrivial) linear maps. Space of bounded linear maps $B(X)$.

Prop^{n}: bounded $\iff$ continuous

Rm^{k}: $\exists$ discontinuous linear maps! (if $\dim(X) = \infty$)

Operator norm $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| = \inf_{C>0} \{(*) \text{ holds}\}$

Prop^{n}: The operator norm is a norm. Moreover, $\|Tx\| \leq \|T\| \cdot \|x\|$ and $\|TS\| \leq \|T\| \, \|S\|$

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Th^m If Y is complete, then $B(X,Y)$ is complete (Banach), but the norm doesn't come from any inner product.

Proof: T_n Cauchy in $B(X,Y)$ $\quad (\|T_n - T_m\| \to 0)$

$n, m \geq N \Rightarrow \|T_n - T_m\| \leq \varepsilon$.

$x \in X \Rightarrow \|T_n x - T_m x\| \leq \|T_n - T_m\| \, \|x\| \leq \varepsilon \|x\|$
$\Rightarrow (T_n x)$ Cauchy $\Rightarrow T_n x \to Tx$.

$\lim_{m \to \infty} \|T_n x - T_m x\| = \|T_n x - Tx\| \leq \varepsilon \|x\|$
$\Rightarrow T_n - T$ bounded $\Rightarrow T$ bounded and
$\|T_n - T\| \leq \varepsilon$ so $T_n \to T$ in norm $\square$

Example: $T : X \to X$: $Tx = \langle x, e \rangle f \quad (e, f \in H)$
bounded, since $\|Tx\|^2 = \langle \langle x, e \rangle f, \langle x, e \rangle f \rangle$
$= |\langle x, e \rangle|^2 \|f\|^2$
$\leq \|x\|^2 \, \|e\|^2 \, \|f\|^2$

hence $\|T\| \leq \|e\| \cdot \|f\|$ and $Te = \|e\|^2 f \Rightarrow \|Te\| / \|e\| = \|e\| \|f\|$.

Dual spaces

ground field

$$X^* = \{ f : X \to \mathbb{C} \mid f \text{ is bounded and linear} \}$$

$\boxed{|f(x)| \leq C \|x\|}$ Continuous linear functionals.

$\dim(X) < \infty$ Let $\{e_1, \dots, e_n\}$ be a basis.

Dual basis $\boxed{\varepsilon^i(e_j) = \delta^i_j}$ $\quad$ ($\varepsilon^i : X \to \mathbb{C}$ linear and bounded since $\dim(X) < \infty$)

Th^m $\{ \varepsilon^i : i = 1, \dots, n \}$ is a basis for X^*

$\Rightarrow \dim(X^*) = \dim(X) = n \Rightarrow X^* \simeq X$.

Proof: Exercise $\square$

Th^m: $X \hookrightarrow X^{**}$ via the map $x \mapsto \mathcal{J}(x)$
$\mathcal{J}(x)(f) = f(x)$

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$\mathcal{J}(x)\ (f) = f(x)$.

Proof $\mathcal{J}$ is linear and $\mathcal{J}(x) = 0 \implies f(x) = 0 \quad \forall f \in H^*$
$\implies f = 0$. (hahn-banach)

- Moreover, $\mathcal{J}$ is an isometry (again using Hahn-Banach).
- X is called reflexive if $\mathcal{J}$ is surjective.

Examples :
- $\dim(X) < \infty \implies X$ reflexive (rank-nullity)
- hilbert spaces are reflexive
- ℓ^p $(1 < p < \infty)$ is reflexive
- ℓ^1, ℓ^∞ not reflexive. (relation with separability)

Th^m : (Riesz - Fréchet) given $y \in X$: $f_y(x) = \langle x, y \rangle$ has $f_y \in X^*$. $(X \hookrightarrow X^*)$. Moreover, $\|y\| = \|f_y\|$.
$\forall f \in X^*\ \exists!\ y \in X : f(x) = \langle x, y \rangle$. Thus, $\boxed{X \simeq X^*}$.

Proof : $|f_y(x)| \leq \|x\| \cdot \|y\|$ and $f_y(y) = \|y\|^2 \implies \|f_y\| = \|y\|$

$f \in X^* \implies \ker(f) \subseteq X$ closed. (Assume $f \neq 0$ so $\ker(f) \neq X$).
$X = \ker(f) \oplus \ker(f)^\perp$ Take $u \in \ker(f)^\perp$, $u \neq 0$.

Define $y = \overline{f(u)} \dfrac{1}{\|u\|^2} u$

- if $x \in \ker(f)$, then $f(x) = 0$ and $\langle x, y \rangle = 0$
- if $x \in \operatorname{span}\{u\}$ then $x = \alpha u$, $f(x) = \alpha f(u)$,
and $\langle x, y \rangle = \alpha \langle u, \overline{f(u)} \frac{1}{\|u\|^2} u \rangle = \alpha \cancel{\frac{1}{\|u\|^2}} f(u) \cancel{\|u\|^2} = \alpha f(u)$

They span since $\forall x \in X$: $x = \underbrace{\left(x - \frac{f(x)}{f(u)} u\right)}_{\in \ker(f)} + \underbrace{\left(\frac{f(x)}{f(u)} u\right)}_{\in \operatorname{span}\{u\}}$ $(f(u) \neq 0)$

[uniqueness] $f(x) = \langle x, y \rangle = \langle x, y' \rangle \quad \forall x \in X$
$\implies \|y - y'\|^2 = \langle y - y', y - y' \rangle = \langle y - y', y \rangle - \langle y - y', y' \rangle$
$= f(y - y') - f(y - y') = 0$ □

Weak Topology.

X - normed linear space. X^* dual.
$\exists$ weakest topology on X such that all $f \in X^*$ are continuous w.r.t. $(\sigma(X, X^*), |\cdot|)$.

Consider $0 \in X$. $\forall f \in X^*$, $f(0) = 0$. Continuity implies

Consider $0 \in X$. $\forall f \in X^*$, $f(0) = 0$. Continuity implies

$f^{-1}(-\varepsilon, \varepsilon)$ is an open neighbourhood of $0 \in X$.

$\Rightarrow U = \{ x \in X : |f(x)| < \varepsilon \}$ is weakly open.

Basis of neighbourhoods $\bigcap_{i=1}^{n} \{ x \in X : |f_i(x - x_0)| < \varepsilon_i \}$

with $x_0 \in X$, $\varepsilon_i > 0$, $f_i \in X^*$.

(may replace ε_i by a single ε).

Th^m: - Weakly open $\Rightarrow$ open
- weakly open $\Leftrightarrow$ $\forall x_0 \in U$ $\exists f_1, \dots, f_n \in X^*$ and $\varepsilon > 0$ such that $\bigcap_{i=1}^{n} \{ x \in X : |f_i(x - x_0)| < \varepsilon \} \subseteq U$.

Weak* topology on X^*

$\sigma(X^*, X)$ - weakest topology making all point evaluations

$\mu_x : X^* \to \mathbb{K}$, $f \mapsto f(x)$ continuous.

Th^m (Alaoglu) X Banach $\Rightarrow$ ball(X^*) is weak*-cpt

- optimization
- X reflexive $\Rightarrow$ ball(X^{**}) is weakly cpt.
$\Rightarrow$ ball(X) — " —.

Weak / Weak* -convergence

$x_n \xrightarrow{w} x$ if x_n is eventually in all wk. nbhds of X.

$\Leftrightarrow f(x_n) \to f(x)$ $\forall f \in X^*$.

Th^m: - Weak limits are unique (Hausdorff property)
- Weakly convergent $\Rightarrow$ bounded.

Proof: - $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$. $\exists f \in X^* : f(x) \neq f(y)$ by Hahn-banach. $f(x_n) \to f(x)$ and $f(x_n) \to f(y)$ $\lightning$

- $\forall f \in X^*$ $\sup_n |f(x_n)| < \infty$ $\Rightarrow \sup_n |\partial(x_n)(f)| < \infty$

$\overset{UBP}{\Rightarrow} \sup_n \|\partial(x_n)\| = \sup_n \|x_n\| < \infty$

$\Rightarrow$ bounded. $\square$

$\underline{Th^m}$ In a Hilbert space $x_n \xrightarrow{w} x \iff \forall y \in X \ \langle x_n, y\rangle \to \langle x, y\rangle$

Proof: Exercise $\square$

(Orthonormal) bases

$\underline{Th^m}$: X Banach space, $\dim(X) = \infty \Rightarrow \{e_1, e_2, \dots\} = E$ can never be a (Hamel) basis for X. ($\operatorname{span} E \neq X$).

Proof: $X = \bigcup_{n=1}^{\infty} \operatorname{span}\{e_1, \dots, e_n\} \overset{\text{Baire}}{\Longrightarrow} X = \operatorname{span}\{e_1, \dots, e_N\}$ ↯

Schauder Basis $\boxed{\overline{\operatorname{span}}\, E = X.}$ $\left(\overline{\operatorname{span} E} = \bigcap\{F \supset E : F \text{ closed lin.}\}\right)$

For Hilbert spaces: Let $E = \{e_1, e_2, \dots\}$ be an ONS

$\langle e_i, e_j\rangle = \delta_{ij}$

$\underline{Th^m}$ (Bessel) $\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|$.

↳ Converges absolutely.

$\underline{Def^n}$ ONB $\iff \overline{\operatorname{span}}\{e_1, e_2, \dots\} = X$.

"basis"

$\underline{Th^m}$: $\iff \{e_i : i \in \mathbb{N}\}^{\perp} = \{0\}$

$\iff \forall x \in X : \boxed{x = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i}$ Fourier series!

$\iff \forall x \in X : \|x\|^2 = \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2$

Example (ℓ^2) $e_i = (0, 0, \dots, 0, 1, 0, \dots)$ is an ONB.

$\uparrow$ ith position

Proof: $x = (x_1, x_2, \dots) \in \ell^2$ and $x \in \{e_i : i \in \mathbb{N}\}^{\perp}$

$\Rightarrow \langle x, e_i\rangle = 0 \ \forall i \Rightarrow x_i = 0 \ \forall i \Rightarrow x = 0$ $\square$

Invertibility

Let X be a Hilbert space, $B(X)$ the algebra of bounded linear operators from X to itself. $B(X)$ has a unit I.

$\underline{Def^n}$: $T \in B(X)$ is invertible (in $B(X)$) $\iff \exists S \in B(X) : \boxed{TS = ST = I}$.

$\underline{Rm^k}$: invertible $\not\Leftarrow\!\!\Rightarrow$ injective and surjective

need to check both if $\dim(X) = \infty$

Example $X = s$ (finitely supported seq^{ns})

both if $\dim(X)=\infty$

<u>Example</u> $X = s$ (finitely supported seq^{ns})

$$T(x_1, x_2, \dots) = (x_1, \tfrac{1}{2}x_2, \dots, \tfrac{1}{n}x_n, \dots) \in B(X)$$
$$S(x_1, x_2, \dots) = (x_1, 2x_2, \dots, nx_n, \dots) \notin B(X).$$

<u>Th^m</u>: If X is <u>Hilbert</u>, then bijective $\Rightarrow$ invertible
$\boxed{\text{OPEN MAPPING}}$.

<u>Proof</u>: $T \in B(X)$ <u>surjective</u> $\Longrightarrow$ T is an open map
T <u>injective</u> $\Longrightarrow$ $T^{-1}: \operatorname{ran}(T) \longrightarrow X$ exists and is linear.
U open, $U \subseteq X$ $\Longrightarrow$ $T(U)$ open. $T(U) = (T^{-1})^{-1}(U)$ □

<u>Example</u>
- s is not Banach.
- $T: \ell^2 \to \ell^2$
  $T(x_1, x_2, \dots) = (x_1, \tfrac{1}{2}x_2, \dots)$ doesn't have closed range. ($T: \ell^2 \to \operatorname{ran}(T) \subseteq \ell^2$.)
- $S: V \subset \ell^2 \to \ell^2$ given by
  $S(x_1, x_2, \dots) = (x_1, 2x_2, \dots)$ with
  $V = \{x \in \ell^2 : Sx \in \ell^2\}$ is <u>not</u> a Closed subspace.
  ↳ Closed graph theorem.

<u>Th^m</u>: $\|T\| < 1 \Rightarrow I - T$ is invertible and $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$.

<u>Proof</u>: $\sum_{n=0}^{\infty} \|T^n\| \leq \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1-\|T\|} < \infty$ and absolute convergence implies convergence. Finish by telescoping series □

<u>Th^m</u> The set of invertible elements is <u>open</u>.

<u>Proof</u>: A invertible, $\|B\|$ small $\Rightarrow$ $A-B$ invertible, since

$$A - B = A - AA^{-1}B = A(I - A^{-1}B) \quad \text{so } \|A^{-1}B\| \text{ small.}$$ □

<u>Spectrum</u> $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ <u>not</u> invertible}\}$

<u>General properties</u> if T is bounded and X is Banach:

- $\sigma(T)$ is closed
- $\sigma(T)$ is bounded ($\lambda \in \sigma(T) \Rightarrow |\lambda| \leq \|T\|$)
- $\sigma(T)$ is nonempty

<u>How can $T - \lambda I$ fail to be invertible?</u>

- $T - \lambda I$ not injective $\Leftrightarrow \ker(T - \lambda I) \neq \{0\}$
  $\Leftrightarrow \lambda$ is an eigenvalue of T
  $\sigma_p(T)$: eigenvalues / point spectrum.
- $T - \lambda I$ injective and $\operatorname{ran}(T - \lambda I)$ not dense

$\sigma_p(T)$: eigenvalues / point spectrum.

- $T - \lambda I$ injective and ran $(T-\lambda I)$ not dense
  ↳ residual spectrum (doesn't exist for SA operators)
- $T - \lambda I$ not bounded below ($\nexists c : \|(T-\lambda I)x\| \geq c\|x\|$).
  ↳ approximate point spectrum (in particular for eigenvalues)

Th^m T is invertible $\Leftrightarrow$ ran (T) dense and T bounded below.

Proof: ($\Rightarrow$) obvious

($\Leftarrow$) ran (T) dense and $\|Tx\| \geq c\|x\|$ implies that T is injective. ($Tx = 0 \Rightarrow \|x\| \leq 0$).
$\Rightarrow T^{-1} : \text{ran}(T) \to X$ exists and is bounded.
Extends uniquely to a bounded inverse on $\overline{\text{ran}(T)} = X$. $\square$